laplace transforms - II

1/12/22 ## Laplace transform of derivatives :

theorem :

If $f(t)$ is continuous for all $t \geq 0$ and of exponential order a as $t \to \infty$ and if $f'(t)$ is of class A, then the laplace transform of $f'(t)$ exists when $p > a$ and $L[f'(t)] = p\bar{f}(p) - f(0)$

where ~~$L[f(t)] = L[f$~~

$$L[f(t)] = \bar{f}(p)$$

proof :

$$L[f'(t)] = \int_0^\infty e^{-pt} f'(t)\, dt$$

Integration ki Differentiation
on rendu vastea f(t) ne
vastamu.

$$= \left[e^{-pt} f(t)\right]_0^\infty - \int_0^\infty -pe^{-pt} f(t)\, dt$$

$$= 0 - f(0) + p\int_0^\infty e^{-pt} f(t)\, dt$$

[$\because$ f(t) is exponential order]

$$= -f(0) + pL[f(t)]$$

$$L[f'(t)] = -f(0) + p\bar{f}(p)$$

$$\therefore L[f'(t)] = p\bar{f}(p) - f(0)$$

prove that $L[f''(t)] = p^2\bar{f}(p) - pf(0) - f'(0)$

$$L[f''(t)] = p^2\bar{f}(p) - pf(0) - f'(0)$$

$$L[f''(t)] = L[f'(t)']$$

$$= pL[f'(t)] - f'(0)$$

$$= p[p\bar{f}(p) - f(0)] - f'(0)$$

$$= p^2\bar{f}(p) - pf(0) - f'(0)$$

Note point :

$$L[f^{(n)}(t)] = p^n\bar{f}(p) - p^{n-1}f(0) - p^{n-2}f(0) - \cdots f^{n-1}(0)$$

problems :—

P Show that the following by using the theorem of transforms of derivatives

1. $L[t] = \frac{1}{p^2}$

2. $L[e^{at}] = \frac{1}{p-a}$

3. $L[-a\sin at] = \frac{-a^2}{p^2+a^2}$

We have $L[f'(t)] = pL[f(t)] - f(0) \rightarrow ①$

1. Here $L[t] = \frac{1}{p^2}$

$f(t) = t \Rightarrow f'(t) = 1$ $\quad f'(t)\frac{dt}{dt} = 1$

$f(0) = 0$

$L[1] = pL[t] - 0$

$\frac{1}{p} = pL[t]$

$\frac{1}{p^2} = L[t]$

$\therefore L[t] = \frac{1}{p^2}$

2. Here $L[e^{at}] = \frac{1}{p-a}$

$L[f'(t)] = pL[f(t)] - f(0)$

$f(t) = e^{at} \Rightarrow f'(t) = ae^{at}$ $\quad f(t) = e^{at}$

$f(0) = 1$ $\quad = e^{a(0)}$

$L[ae^{at}] = pL[e^{at}] - 1$ $\quad = e^0$

$1 = pL[e^{at}] - a\cdot L[e^{at}]$ $\quad f(0) = 1$

$1 = (p-a)L[e^{at}]$

$\frac{1}{p-a} = L[e^{at}]$

$\therefore L[e^{at}] = \frac{1}{p-a}$

3. Here $L[-a\sin at] = \frac{-a^2}{p^2+a^2}$

$L[f''(t)] = p^2L[f(t)] - pf(0) - f'(0)$

$f'(t) = -a\sin at \Rightarrow f(0) = 0$

$f'(t) = -a^2\cos at \Rightarrow f'(0) = -a^2$

$f''(t) = a^3\sin at$

$L[a^3\sin at] = p^2 L[a^2\cos at] - p(0) + a^2$

$L[a^3\sin at] = p^2 L[-a\sin at] - p(0) + a^2$

$(-a^2) L[(-a)\sin at] - p^2 L[-a\sin at] = a^2$

$-L[-a\sin at][a^2+p^2] = +a^2$

$$L[-a\sin at] = \frac{-a^2}{p^2+a^2}$$

$$\therefore L[-a\sin at] = \frac{-a^2}{p^2+a^2}$$

Initial value theorem: —

1(a) If $f(t)$ is piece wise continuous for all $t \geq 0$ and of exponential order as $t \to \infty$ or $f'(t)$ is of class A then ~~$\lim_{t\to 0} f(t) = \lim_{p\to\infty}$~~

$$\lim_{t\to 0} f(t) = \lim_{p\to\infty} pL[f(t)] = \lim_{p\to\infty} (p\bar{f}(p))$$

Proof : —

By the theorem on laplace transform of derivatives, we have

$$L[f'(t)] = \int_0^\infty e^{-pt} f'(t)\,dt$$

$$= pL[f(t)] - f(0) \to (1)$$

Since $f'(t)$ is piece wise continuous and of exponential order

$$\lim_{p\to\infty} \int_0^\infty e^{-pt} f'(t)\,dt = 0 \quad \left[\because \lim_{p\to\infty} \int_0^\infty e^{-pt} f'(t)\,dt\right.$$

$$= \int_0^\infty \left(\lim_{p\to\infty} e^{-pt}\right) f'(t)\,dt$$

$$= \int_0^\infty 0\, f'(t)\,dt$$

$$= 0$$

Taking $p\to\infty$ in equation ① we have

$$0 = \lim_{p\to\infty} pL[f(t)-f(0)]$$

$$f(0) = \lim_{p\to\infty} pL[f(t)]$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{p\to\infty} pL[f(t)] = \lim_{p\to\infty} pf(p).$$

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Q Evaluate $\int_0 \int_0 \frac{r\,d\theta\,dr}{\sqrt{a^2-r^2}}$

Final value theorem:—

If $f(t)$ is continuous for all $t \geq 0$ and of exponential order as $t\to\infty$ or $f'(t)$ is of class A then

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pL[f(t)] = \lim_{p\to 0} [pf(p)]$$

Proof:—

By the theorem on laplace transforms of derivatives: we have

$$L[f'(t)] = \int_0^\infty e^{-pt} f'(t)\,dt$$

$$= pL[f(t)] - f(0) \to ①$$

since $f'(t)$ is piece wise continuous and of exponential order.

$$\lim_{p\to 0} \int_0^\infty e^{-pt} f'(t)\,dt$$

$$\therefore \lim_{p\to 0} \int_0^\infty e^{-pt} f'(t)\,dt$$

$$= \int_0^\infty (\lim_{p\to 0} e^{-pt}) f'(t)\,dt$$

$$= \int_0^\infty 0 \cdot f'(t)\,dt = 0$$

Taking $p\to 0$ in equation ① we have

$$0 = \lim_{p\to 0} pL[f(t)] - f(0)$$

$$f(0) = \lim_{p\to 0} pL[f(t)]$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{p\to 0} pL[f(t)]$$

$$= \lim_{p\to 0} pf(p)$$

problems :—

P Imp If $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}\, e^{-1/4p}}{2p^{3/2}}$ then find $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

Given $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}\, e^{-1/4p}}{2p^{3/2}}$

$f(t) = \sin\sqrt{t} \Rightarrow f(0) = 0$

$f'(t) = \frac{1}{2\sqrt{t}}\cos\sqrt{t} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ also

$L[f'(t)] = pL[f(t) - f(0)]$

$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = pL[\sin\sqrt{t}] - 0$

$\frac{1}{\cancel{2}} L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = p\,\frac{\sqrt{\pi}\, e^{-1/4p}}{\cancel{2}p^{3/2}}$

$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = p\,\frac{\sqrt{\pi}\, e^{-1/4p}}{p^{3/2}}$

$= \frac{\cancel{p}\sqrt{\pi}\, e^{-1/4p}}{\cancel{p}\, p^{1/2}}$ $\qquad p^{3/2} = p^1 p^{1/2}$

$= \sqrt{\frac{\pi}{p}}\, e^{-1/4p}$ $\qquad p^{\frac{1}{2}} = \sqrt{p}$

P If $L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{p^{3/2}}$ then show that $L\left[\frac{1}{\sqrt{t\pi}}\right] = \frac{1}{\sqrt{p}}$

Given $L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{p^{3/2}}$

$f(t) = 2\sqrt{\frac{t}{\pi}} \Rightarrow f(0) = 0$

$f'(t) = \frac{\cancel{2}}{\sqrt{\pi}}\,\frac{1}{\cancel{2}\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$

$L[f'(t)] = pL[f(t)] - f(0)$

$L\left[\frac{1}{\sqrt{\pi t}}\right] = pL\left[2\sqrt{\frac{t}{\pi}}\right] - 0$

$L\left[\frac{1}{\sqrt{\pi t}}\right] = p\,\frac{1}{p^{3/2}}$

$$L\left[\frac{1}{\sqrt{t\pi}}\right] = \cancel{p}\,\frac{1}{p^{1/2}\cancel{p}}$$ $$\frac{1}{p^{3/2}} = \frac{1}{p^{1/2}p^{1}}$$

$$L\left[\frac{1}{\sqrt{t\pi}}\right] = \frac{1}{p^{1/2}}$$

$$\therefore L\left[\frac{1}{\sqrt{t\pi}}\right] = \frac{1}{\sqrt{p}}$$

Transforms of integrals :—

Statement :—

If $f(t)$ is piece wise continuous and of exponential order then the laplace transform of $\int_0^t f(t)dt$ is $\frac{1}{p}L[f(t)]$

i.e., $L\left[\int_0^t f(t)dt\right] = \frac{1}{p}L[f(t)] = \frac{1}{p}f(p)$.

Note point :—

If $L[f(t)] = \frac{1}{p}f(p)$ then $L\left[\int_0^t\int_0^t f(t)dt\right] = \frac{1}{p^2}f(p)$

problems :—

P Find the laplace transform of $\int_0^t e^{-t}\cos t\,dt$.

$$= \int_0^t e^{-t}\cos t\,dt$$

$f(t) = \cos t$

$$L[f(t)] = L[\cos t]$$

$$= \frac{p}{p^2+1} = f(p)$$

$$L[e^{-t}\cos t] = \frac{p+1}{(p+1)^2+1} = f(p)$$ by First shifting theorem

$$L\left[\int_0^t e^{-t}\cos t\right]dt = \frac{1}{p}\times\frac{p+1}{(p+1)^2+1}$$ $$\frac{p}{p^2+1}$$

$$= \frac{1}{p}\times\frac{p+1}{p^2+1+2p+1}$$

$$= \frac{1}{p}\times\frac{p+1}{p^2+2p+2}$$

$= \frac{p+1}{p^2+2p+2}$

$L\left\{\int_0^t e^{-t}\cos t\, dt\right\} = \frac{p+1}{3p+2}$

P Find the laplace transform of $L\left[\int_0^t \sin t\, dt\right]$

$f(t) = \sin t$

$L[f(t)] = L[\sin t] \qquad = \frac{a}{p^2+a^2}$

$= \frac{1}{p^2+1}$

By T of I $L\left[\int_0^t \sin t\, dt\right] = \frac{1}{p} \cdot \frac{1}{p^2+1}$

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P Find the laplace transform of $\int_0^t e^{-t} \sinh t\, dt$.

$\int_0^t e^{-t} \sinh t\, dt$

$\sinh at = \frac{a}{p^2-a^2}$

$f(t) = \sinh t$

$L[f(t)] = L[\sinh t]$

$= \frac{1}{p^2-1}$

$= f(p)$

By First shifting theorem. $(p+1)$ in p place.

$L[e^{-t} \sinh t] = \frac{1}{(p+1)^2-1} = f(p)$

By transforms of integrals

$L\left[\int_0^t e^{-t} \sinh t\, dt\right] = \frac{1}{p} f(p)$

$= \frac{1}{p} \frac{1}{(p+1)^2-1}$

$= \frac{1}{p} \frac{1}{p^2+2p}$

$= \frac{1}{p^2(p+2)}$

P Find the laplace transform of $\int_0^t\int_0^t \cosh at\, dt$.

$$\int_0^t\int_0^t \cosh at \, dt$$

$$f(t) = \cosh at$$

$$L[f(t)] = \frac{p}{p^2-a^2} = f(p)$$

By transform of integral.

$$L\left[\int_0^t \cosh at\, dt\right] = \frac{1}{\cancel{p}}\frac{\cancel{p}}{p^2-a^2}$$

$$= \frac{1}{p^2-a^2}$$

$$= f(p)$$

or $\frac{1}{p}\frac{p}{p^2-a^2}$

$= f(p)$

By T. of I

$$L\left[\int_0^t\int_0^t \cosh at\, dt\, dt\right]$$

$$= \frac{1}{p}\frac{1}{p^2-a^2}$$

By T. of I

$$L\left[\int_0^t\int_0^t \cosh at\, dt\, dt\right]$$

$$= \frac{1}{p\cancel{^2}}\frac{\cancel{p}}{p^2-a^2}$$

$$= \frac{1}{p}\frac{1}{p^2-a^2}$$

Derivative of integral function :-

In finding the derivative of an integral, it is not always possible to evaluate the first the integral and then derivative. In that cases we may reversing the order of integration and differentiation for that we use the following rule.

Lebnitz's Rule :-

If $f(x,y)$ and $\frac{d}{dx}[f(x,t)]$ are continuous functions in x,t then $\frac{d}{dx}\left[\int_a^b f(x,t)\,dx\right] = \int_a^b \frac{d}{dx}[f(x,t)]\,dx$

Laplace transform of $t^n f(t)$:- (Multiplication by t)

Imp theorem :—

statement :—

(a) If $L[f(t)] = f(p)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{dp^n}[f(p)]$, $n = 1, 2, 3 \ldots\ldots$

proof :—

We prove

$$L[t^n f(t)] = (-1)^n \frac{d^n}{dp^n}[f(p)], \quad n = 1, 2, 3 \ldots\ldots \rightarrow (1)$$

By induction on n

$$f(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

$$\frac{d}{dp} f(p) = \frac{d}{dp} \int_0^\infty e^{-pt} f(t)\, dt$$

By leibnitz's rule

$$= \int_0^\infty \frac{d}{dp}[e^{-pt} f(t)]\, dt.$$

$$= \int_0^\infty e^{-pt} (-t) f(t)\, dt$$

$$= -\int_0^\infty e^{-pt} t\, f(t)\, dt$$

$$\frac{d}{dp} f(p) = -L[t f(t)] \qquad L[f(t)] = \int_0^\infty e^{-pt} f(t)\, dt$$

$$\therefore +L[t f(t)] = -\frac{d}{dp} f(p)$$

$\therefore$ 1 is true for $n = 1$

Assume that 1 is true for $n = m$

$$\therefore L[t^m f(t)] = (-1)^m \frac{d^m}{dp^m} f(p)$$

Now $\frac{d}{dp}[L[t^m f(t)]] = (-1)^m \frac{d^m}{dp^m} \frac{d}{dp} f(p)$

$$\frac{d}{dp}[L[t^m f(t)]] = (-1)^m \frac{d^{m+1}}{dp^{m+1}} f(p)$$

$$\Rightarrow \frac{d}{dp}\left[\int_0^\infty e^{-pt} t^m f(t)\,dt\right] = (-1)^m \frac{d^{m+1}}{dp^{m+1}} f(p)$$

By Leibnitz's rule

$$\Rightarrow \int_0^\infty \frac{d}{dp}\left[e^{-pt} t^m f(t)\,dt\right] = (-1)^m \frac{d^{m+1}}{dp^{m+1}}[f(p)]$$

$$\Rightarrow \int_0^\infty e^{-pt}(-t) t^m f(t)\,dt = (-1)^m \frac{d^{m+1}}{dp^{m+1}}[f(p)]$$

$$\Rightarrow -\int_0^\infty e^{-pt} t^{m+1} f(t)\,dt = (-1)^m \frac{d^{m+1}}{dp^{m+1}}[f(p)]$$

$$-L[t^{m+1} f(t)] = (-1)^m \frac{d^{m+1}}{dp^{m+1}}[f(p)]$$

$$L[t^{m+1} f(t)] = (-1)(-1)^m \frac{d^{m+1}}{dp^{m+1}}[f(p)]$$

$$L[t^{m+1} f(t)] = (-1)^{m+1} \frac{d^{m+1}}{dp^{m+1}}[f(p)]$$

$\therefore$ 1 is true for $n = m+1$.

$\therefore$ By mathematical induction $L[t^n f(t)] = (-1)^n \frac{d^n}{dp^n} f(p)$ $\forall n$

Note point:

If $L[f(t)]$ then $L[t f(t)] = \frac{-d}{dp}[f(p)]$

$= -f'(p)$.

P. Evaluate $L[t\cos at]$

$L[t\cos at]$

$f(t) = \cos at$

$L[f(t)] = L[\cos at] = \frac{p}{p^2+a^2} = f(p)$

$L[t f(t)] = \frac{-d}{dp} f(p)$

$L[t\cos at] = \frac{-d}{dp}\left[\frac{p}{p^2+a^2}\right]$

$$= -\left[\frac{(p^2+a^2)(1) - p(2p)}{(p^2+a^2)^2}\right]$$ ← Suv formula.

$$= -\left[\frac{p^2+a^2-2p^2}{(p^2+a^2)^2}\right]$$

$$= -\left[\frac{-p^2+a^2}{(p^2+a^2)^2}\right]$$

$$= \frac{p^2-a^2}{[p^2+a^2]^2}$$

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Evaluate $L[\sin at - at\cos at]$

$$L[\sin at - at\cos at] = L[\sin at] - L[at\cos at]$$

$$= L[\sin at] - aL[t\cos at]$$

$$= \frac{a}{p^2+a^2} - \frac{a(p^2-a^2)}{(p^2+a^2)^2}$$

$$= \frac{a(p^2+a^2) - ap^2 + a^3}{(p^2+a^2)^2} \Rightarrow \frac{ap^2 + ap^2 - ap^2 + a^3}{(p^2+a^2)^2}$$

$$= \frac{2a^3}{(p^2+a^2)^2}$$

Find $L[t^2\cos at]$

$$(-1)^n \frac{d^{n+1}}{dx^n}[f(p)]$$

$$L[t^2\cos at] = (-1)^2 \frac{d^2}{dx^2}[f(p)]$$

$$f(t) = \cos at$$

$$L[f(t)] = L[\cos at]$$

$$= \frac{p}{p^2+a^2} = f(p)$$

$$L[t^2\cos at] = \frac{d^2}{dp^2}\left[\frac{p}{p^2+a^2}\right]$$

$$= \frac{d}{dp}\left[\frac{d}{dp}\left[\frac{p}{p^2+a^2}\right]\right]$$

$$= \frac{d}{dp}\left(\frac{p^2+a^2(1) - p(2p)}{(p^2+a^2)^2}\right)$$

$$= \frac{d}{dp}\left[\frac{a^2-p^2}{(p^2+a^2)^2}\right]$$

$$= \frac{(p^2+a^2)^2(-2p) - (a^2-p^2)\,2(p^2+a^2)\,2p}{(p^2+a^2)^4}$$

$$= \frac{(p^2+a^2)\left[-2p\,(p^2+a^2) - 4p\,(a^2-p^2)\right]}{(p^2+a^2)^{\cancel{4}\,3}}$$

$$= \frac{-2p^3 - 2pa^2 - 4pa^2 + 4p^3}{(p^2+a^2)^3}$$

$$= \frac{2p^3 - 6pa^2}{(p^2+a^2)^3}$$

Find $L\{t^2\cos 3t\}$.

P Find $L[(t^2-3t+2)\sin 3t]$

$L[(t^2-3t+2)\sin 3t] = L[t^2\sin 3t] - 3L[t\sin 3t] + 2L[\sin 3t]$

$L[t^2\sin 3t] = (-1)^2 \frac{d^2}{dp^2}[f(p)]$ $\quad = (-1)^n \frac{d^n}{dp^n}[f(p)]$ ①

$f(t) = \sin 3t = L[f(t)] = L[\sin 3t]$

$= \frac{3}{p^2+9}$ $\quad \frac{a}{p^2+a^2}$

$= f(p)$

$L[t^2\sin 3t] = \frac{d^2}{dp^2}\left(\frac{3}{(p^2+9)}\right)$

$= \frac{d}{dp}\left[\frac{d}{dp}\left[\frac{3}{p^2+9}\right]\right]$

$= \frac{d}{dp}\left[\frac{(p^2+9)(0) - 3(2p)}{(p^2+9)^2}\right]$

$= \frac{d}{dp}\left[\frac{-6p}{(p^2+9)^2}\right]$

$= \frac{(p^2+9)^2(-6) + (6p)\,2(p^2+9)(2p)}{(p^2+9)^4}$

$= \frac{-6(p^2+9)^2 + 24p^2(p^2+9)}{(p^2+9)^4}$

$= \frac{\cancel{(p^2+9)}[-6p^2-54+24p^2]}{(p^2+9)^4}$

$= \frac{18p^2-54}{(p^2+9)^3}$

$L[(t^2-3t+2)\sin 3t] = \frac{18p^2-54}{(p^2+9)^3} + \frac{18p}{(p^2+9)^2} + \frac{6}{p^2+9}$

P Find $L[te^{3t}\sin 2t]$

$f(t) = \sin 2t$

$L[f(t)] = L[\sin 2t] = \frac{2}{p^2+4}$ $\quad \frac{a}{p^2+a^2}$

By first shifting theorem

$L[e^{3t}\sin 2t] = \frac{2}{(p-3)^2+4}$

$= \frac{2}{p^2+9-6p+4}$

$= \frac{2}{p^2-6p+13} = f(p)$

$L[te^{3t}\sin 2t] = (-1)\frac{d}{dp}(f(p))$

$= (-1)\frac{d}{dp}\left[\frac{2}{p^2-6p+13}\right]$

$= -\frac{\left[(p^2-6p+13)(0)-2(2p-6)\right]}{(p^2-6p+13)^2}$

$= \frac{4p-12}{(p^2-6p+13)^2}$

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Division by t :—

If $L[f(t)] = f(p)$ then $L\left[\frac{f(t)}{t}\right] = \int_p^\infty f(p)\,dp$

problems :—

P Find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

$L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

$f(t) = e^{-at}-e^{-bt}$

$L[f(t)] = L[e^{-at}-e^{-bt}]$

$= L[e^{-at}] - L[e^{-bt}]$ $\qquad L[e^{at}] = \frac{1}{p-a}$

$= \frac{1}{p+a} - \frac{1}{p+b} = f(p)$

$L\left[\frac{f(t)}{t}\right] = L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_p^\infty f(p)\,dp$

$= \int_p^\infty \left[\frac{1}{p+a} - \frac{1}{p+b}\right] dp$

$= \int_p^\infty \frac{1}{p+a}\,dp - \int_p^\infty \frac{1}{p+b}\,dp$

$$= \left[\log(p+a) - \log(p+b)\right]_p^\infty$$

$$= \left[\log\left[\frac{p+a}{p+b}\right]\right]_p^\infty$$

$$= \log\left[\frac{p+a}{p+b}\right]$$

Find $L\left[\frac{\sinh at}{t}\right]$

$$L\left[\frac{\sinh at}{t}\right]$$

$$f(t) = \sinh at$$

$$L[f(t)] = L[\sinh at]$$

$$= \frac{a}{p^2-a^2}$$

$$= \frac{a}{p^2-a^2} = f(p)$$

$$L\left[\frac{f(t)}{t}\right] = L\left[\frac{\sinh at}{t}\right] = \int_p^\infty f(p)\,dp$$

$$= \int_p^\infty \left[\frac{a}{p^2-a^2}\right] dp$$

$$= a \cdot \frac{1}{2a}\left[\log\left[\frac{p-a}{p+a}\right]\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\left[\frac{1-\frac{a}{p}}{1+\frac{a}{p}}\right]\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\left[\frac{1-0}{1+0}\right] - \log\left[\frac{1-\frac{a}{p}}{1+\frac{a}{p}}\right]\right]$$

$$= -\frac{1}{2}\log\left[\frac{p-a}{p+a}\right]$$

$$= \frac{1}{2}\log\left[\frac{p+a}{p-a}\right]$$

P Find $L\left[\frac{e^{-3t}\sin 2t}{t}\right]$

$L\left[\frac{e^{-3t}\sin 2t}{t}\right]$

$f(t) = \sin 2t$ By f.s. theorem $\sin at = \frac{a}{p^2+a^2}$

$L[f(t)] = \frac{2}{p^2+4} = f(p)$

By first shifting theorem we have

$L[e^{-3t}\sin 2t] = f(p+3) = \frac{2}{(p+3)^2+2^2} = f(p)$

By Division by 't' we have

$L\left[\frac{e^{-3t}\sin 2t}{t}\right] = \int_p^\infty f(p)\,dp$

$= \int_p^\infty \frac{2}{(p+3)^2+2^2}\,dp$

$= 2\cdot\frac{1}{2}\left[\tan^{-1}\left(\frac{p+3}{2}\right)\right]_p^\infty$

$= \tan^{-1}\left[\left(\frac{p+3}{2}\right)\right]_p^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}\left[\frac{p+3}{2}\right]$

$= \tan^{-1}\left[\tan\frac{\pi}{2}\right] - \tan^{-1}\left[\frac{p+3}{2}\right]$

$= \frac{\pi}{2} - \tan^{-1}\left[\frac{p+3}{2}\right]$

$= \cot^{-1}\left[\frac{p+3}{2}\right]$

P Find $L\left[\int_0^t \frac{e^{t}\sin t}{t}\,dt\right]$

$L\left[\int_0^t \frac{e^{t}\sin t}{t}\,dt\right]$

$f(t) = \sin t$

$L[f(t)] = L[\sin t]$

$= \frac{1}{p^2+1} = f(p)$

$$L\left[\frac{\sin t}{t}\right] = \int_p^\infty \frac{1}{p^2+1}\,dp$$ by Division by 't'

$$L\left[\frac{f(t)}{t}\right] = \int_p^\infty$$

$$= \left[\tan^{-1}p\right]_p^\infty$$

$$= \left[\tan^{-1}\infty - \tan^{-1}p\right]$$

$$= \frac{\pi}{2} - \tan^{-1}p$$

$L[e^{at}f$

$$= \cot^{-1}p = f(p)$$

By First shifting theorem

$$L\left[\frac{e^{t}\sin t}{t}\right] = \cot^{-1}(p-1) = f(p)$$

By transverse of integrals

$$L\left[\int_0^t \frac{e^{t}\sin t}{t}\,dt\right] = \frac{1}{p}f(p)$$

$$= \frac{1}{p}\cot^{-1}(p-1)$$

Evaluation of Integrals by laplace transforms:

P Evaluate $\int_0^\infty e^{-4t} \sin 3t \, dt$

$$\int_0^\infty e^{-pt} f(t) \, dt$$

$p = 4 \quad f(t) = \sin 3t$

$$L[f(t)] = L[\sin 3t]$$

$$= \frac{3}{p^2+9}$$

$$\int_0^\infty e^{-4t} \sin 3t \, dt$$

$$= \frac{3}{4^2+9} = \frac{3}{16+9}$$

$$= \frac{3}{25}$$

P show that $\int_0^\infty t e^{-3t} \sin t \, dt = \frac{3}{50}$

$p = 3, \quad f(t) = t \sin t$

$$L[f(t)] = L[t \sin t]$$

$$f(t) = \sin t$$

$$L[f(t)] = L[\sin t]$$

$$= \frac{1}{p^2+1}$$

$$L[t \sin t] = (-1) \frac{d}{dp} \left(\frac{1}{p^2+1} \right)$$

$$= -\frac{[(p^2+1)(0) - 1(2p)]}{(p^2+1)^2}$$

$$\int_0^\infty t e^{-3t} \sin t \, dt = \frac{2\times 3}{(10)^2}$$

$$= \frac{\cancel{2}\times 3}{\cancel{10}_{5}\times 10}$$

$$= \frac{3}{50}$$

Q Evaluate $\int_0^\infty \frac{e^{at} - e^{-bt}}{t} dt$

Here $p = 0$, $f(t) =$

Laplace transform of some special function

Definition :—

The error function is denoted by erf(t) and is defined as $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^{t} e^{-x^2} dx$

problems :—

Q prove that $L[\operatorname{erf}(\sqrt{t})] = \frac{1}{p\sqrt{p+1}}$ and hence deduce that

1. $L[e^{3t} \operatorname{erf}(\sqrt{t})] = \frac{1}{(p-3)\sqrt{p-2}}$

2. $L[\operatorname{erf}(2\sqrt{t})] = \frac{2}{p\sqrt{p^2+4}}$

3. $L[t \operatorname{erf}(2\sqrt{t})] = \frac{3p+8}{p^2(p+4)^{3/2}}$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[\sqrt{t} - \frac{(\sqrt{t})^3}{3} + \frac{(\sqrt{t})^5}{5\cdot 2!} - \frac{(\sqrt{t})^7}{7\cdot 3!} + \cdots\right]$$

$$L[\operatorname{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}} \left[L[t^{1/2}] - \frac{L[t^{3/2}]}{3} + \frac{L[t^{5/2}]}{5\cdot 2!} - \frac{L[t^{7/2}]}{7\cdot 3!} + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma 3/2}{p^{3/2}} - \frac{1}{3}\frac{\Gamma 5/2}{p^{5/2}} + \frac{1}{5\cdot 2!}\frac{\Gamma 7/2}{p^{7/2}} - \frac{1}{7\cdot 3!}\frac{\Gamma 9/2}{p^{9/2}} + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2}\cdot\frac{1}{p^{3/2}} - \frac{\sqrt{\pi}}{2}\frac{1}{2}\frac{1}{p^{5/2}} + \frac{\sqrt{\pi}}{2}\frac{1}{2}\cdot\frac{3}{4}\frac{1}{p^{7/2}} - \frac{\sqrt{\pi}}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{p^{9/2}} + \cdots\right]$$

$$= \frac{1}{p^{3/2}} \left[1 - \frac{1}{2}\frac{1}{p} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{p^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{p^3} + \cdots\right]$$

$$= \frac{1}{p^{3/2}} \left[1 + \frac{1}{p}\right]^{-1/2}$$

$$= \frac{1}{p\sqrt{p+1}}$$

1. $L[e^{3t} \operatorname{erf}(\sqrt{t})] = \frac{1}{(p-3)\sqrt{p-2}}$

By F.S.T $L[e^{3t} \operatorname{erf}(\sqrt{t})] = \frac{1}{(p-3)\sqrt{(p-3)+1}}$

$$= \frac{1}{(p-3)\sqrt{p-2}}$$

2. $L[\operatorname{erf}(2\sqrt{t})] = \frac{2}{p\sqrt{p+4}}$

By change of scale property

$$L[\operatorname{erf}(2\sqrt{t})] = L[\operatorname{erf}\sqrt{4t}]$$

$$= \frac{1}{4}\frac{1}{(p/4)\sqrt{p/4+1}}$$

$$= \frac{2}{p\sqrt{p+4}}$$

3. $L[t\,erf(2\sqrt{t})] = \dfrac{3p+8}{p^2(p+4)^{3/2}}$

By Multiple integrals

$$L[t\,erf(2\sqrt{t})] = \frac{-d}{dp}\left[\frac{2}{p\sqrt{p+4}}\right]$$

$$= \frac{3p+8}{p^2(p+4)^{3/2}}$$

23/01/23 **Definition:**

The bessel function of orders 'n' is danoted by $Jn(t)$ and defined as

$$Jn(t) = \frac{t^n}{2^n\Gamma(n+1)}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2\cdot4(2n+2)(2n+4)} \cdots\cdots\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$$

P prove that $L[J_0(t)] = \dfrac{1}{\sqrt{1+p^2}}$

We know that $Jn(t) = \dfrac{t^n}{2^n\Gamma(n+1)}\left[1 - \dfrac{t^2}{2(2n+2)} + \dfrac{t^4}{2\cdot4(2n+2)(2n+4)} \cdots\right]$

put $n=0$

$$L[J_0(t)] = \frac{1}{\sqrt{1+p^2}}$$

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)}\left(\frac{t}{2}\right)^{0+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,r!}\left(\frac{t}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$$

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots\cdots$$

$$L[J_0(t)] = L(1) - \frac{1}{2^2}L[t^2] + \frac{1}{2^2 4^2}L[t^4] - \frac{1}{2^2 4^2 6^2}L[t^6] + \cdots\cdots$$

$$= \frac{1}{p} - \frac{1}{2^2}\frac{2!}{p^3} + \frac{1}{2^2 4^2}\frac{4!}{p^5} - \frac{1}{2^2 4^2 6^2}\frac{6!}{p^7} + \cdots$$

$$= \frac{1}{p}\left[1 - \frac{1}{2}\frac{1}{p^2} + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{p^2}\right)^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{p^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{p}\left[1+\frac{1}{p^2}\right]^{-1/2}$$

$$= \frac{1}{p}\left[\frac{1+p^2}{p^2}\right]^{-1/2}$$

$$= \frac{1}{\sqrt{1+p^2}}$$

P| prove that $L[J_0(at)] = \frac{1}{\sqrt{p^2+a^2}}$

[Jo(t)] problem cheesi edi cheyali

From the above problem $[J_0(t)] = \frac{1}{\sqrt{1+p^2}}$

By change of scala property $J_0(at) = \frac{1}{a} f(p/a)$

$$= \frac{1}{a}\cdot\frac{\cancel{p/a}}{\sqrt{1+\frac{p^2}{a^2}}}$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{1+\frac{p^2}{a^2}}}$$

$$= \frac{1}{\sqrt{a^2+p^2}}$$

P| prove that $L[t\,J_0(at)] = \frac{p}{(p^2+a^2)^{3/2}}$

Before problems anni rasi edi rayali

We know that $L[J_0(at)] = \frac{1}{\sqrt{a^2+p^2}}$

$$L[t\,J_0(at)] = (-1)^1\frac{d}{dp}\left[\frac{1}{\sqrt{p^2+a^2}}\right]$$

$$= \frac{p}{(p^2+a^2)^{3/2}}$$

P| prove that $L[e^{-at}\,J_0(at)] = \frac{1}{\sqrt{p^2+2ap+2a^2}}$

Jo(t) cheyali Jo(at) kuda cheesi cheyali problemki

We know that $L[J_0(at)] = \frac{1}{\sqrt{a^2+p^2}} = f(p)$

By First shifting theorem

$$L[e^{-at}J_0(at)] = \frac{1}{\sqrt{a^2+(p+a)^2}}$$

$$= \frac{1}{\sqrt{a^2+p^2+a^2+2ap}}$$

$$= \frac{1}{\sqrt{p^2+2ap+2a^2}}$$

P prove that $\int_0^\infty J_0(t)\,dt = 1$

Given $\int_0^\infty J_0(t)\,dt = 1$ $J_0(t)$ problem choosi remaining chayali

$$L[J_0(t)] = \frac{1}{\sqrt{1+p^2}}$$

$$\Rightarrow \int_0^\infty J_0(t)\,e^{-pt}\,dt = \frac{1}{\sqrt{1+p^2}}$$

put $p=0$

We get $\int_0^\infty J_0(t)\,dt = 1$

P prove that $L[J_1(t)] = 1 - \frac{p}{\sqrt{p^2+1}}$

We know that $J_0'(t) = -J_1(t)$

From the theorem on laplace transform of derivative, we have $L[f'(t)] = pL[f(t)] - f(0)$

$$\therefore L[J_1(t)] = L[J_0'(t)] = -pL[J_0(t)] - J_0(0)$$

$$= -\left[p\frac{1}{\sqrt{p^2+1}} - 1\right]$$

$$= 1 - \frac{p}{\sqrt{p^2+1}}$$

P prove that $L[t\,J_1(t)] = \frac{1}{(p^2+1)^{3/2}}$

$J_1(t) = f(p)$ before problem rasi remaining answer

We know that $L[J_1(t)] = 1 - \frac{p}{\sqrt{p^2+1}}$

By Multiple integrals.

Laplace transforms of periodic functions:

A function $f(t)$ is said to be a periodic function if for all t, there exist a positive constant 'T' such that $f(t+T) = f(t)$.

the least value $T > 0$, such that $f(t+T) = f(t)$ is called period of $f(t)$.

Eg: — $\sin t$ is a periodic function with period 2π.

theorem : —

If $f(t)$ is a periodic function with period T then $L[f(p)] = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\,dt$.

problems : —

1. Find the laplace transform of the output of a full sinewave rectifier given as

$$f(t) = \begin{cases} E\sin\omega t, & 0 < t < \pi/\omega \\ 0, & t > \pi/\omega \end{cases}$$

We know that $L[f(p)] = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\,dt$

Here the period $T = \pi/\omega$

$$L[f(t)] = \frac{1}{1-e^{-p\pi/\omega}} \left[\int_0^{\pi/\omega} e^{-pt} f(t)\,dt + \int_{\pi/\omega}^{\pi/\omega} e^{-pt} f(t)\,dt \right]$$

$$= \frac{1}{1-e^{-p\pi/\omega}} \left[\int_0^{\pi/\omega} E\, e^{-pt} \sin\omega t\,dt \right]$$

$$= \frac{E}{1-e^{-p\pi/\omega}} \int_0^{\pi/\omega} e^{-pt} \sin\omega t\,dt$$

$$= \frac{E}{1-e^{-p\pi/\omega}} \left[\frac{e^{-pt}}{p^2+\omega^2} \left(-p\sin\omega t - \omega\cos\omega t \right) \right]_0^{\pi/\omega}$$

limitni substitute cheyali

$\cos 180 = -1$

$\sin 180 = 0$

$$= \frac{E}{(1-e^{-p\pi/\omega})(p^2+\omega^2)} \left[e^{-p\pi/\omega}\omega + \omega \right]$$

$$= \frac{E\omega}{(1-e^{-p\pi/\omega})(p^2+\omega^2)} (1+e^{-p\pi/\omega}) \quad \frac{e^{p\pi/2\omega}}{e^{p\pi/2\omega}} \left[\frac{1+e^{-p\pi/\omega}}{1-e^{-p\pi/\omega}} \right]$$

L.C.M

$$= \frac{E\omega}{(p^2+\omega^2)} \left[\frac{e^{p\pi/2\omega} + e^{-p\pi/2\omega}}{e^{p\pi/2\omega} - e^{-p\pi/2\omega}} \right] \qquad \frac{e^{p\pi/2\omega} + e^{p\pi/2\omega}\ \frac{p\pi}{\omega}}{e^{p\pi/2\omega} + e^{-p\pi/2\omega}}$$

$$= \frac{E\omega}{p^2+\omega^2} \cosh(p\pi/2\omega)$$

Q. Find the laplace transform of wave function of period $2a$, defined as $f(t) = \begin{cases} k & \text{when } 0<t<a \\ -k & \text{when } a<t<2a \end{cases}$

We know that $L[f(p)] = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\,dt$

$$L[f(t)] = \int_0^a e^{-pt} f(t)\,dt + \int_a^{2a} e^{-pt} f(t)\,dt$$

$$= \int_0^a e^{-pt}\,k + \int_a^{2a} e^{-pt}(-k)\,dt$$

$$= \frac{1}{1-e^{-2ap}} \left(k \left[\frac{e^{-pt}}{-p}\right]_0^a - k \left[\frac{e^{-pt}}{-p}\right]_a^{2a} \right)$$

$$-\cancel{\frac{k}{p}}$$

$$= \frac{1}{1-e^{-2ap}} \left[-\frac{k}{p}\left[e^{-pa}-1\right] + \frac{k}{p}\left[e^{-2ap} - e^{-ap}\right] \right]$$

$$= \frac{1}{1-e^{-2ap}} \frac{k}{p} (1-e^{-ap})^2$$

$$= \frac{1}{1-(e^{-ap})^2} \frac{k}{p} (1-e^{-ap})^2$$

$$(a+b)^2 = a^2 - b^2$$

$$= \frac{1}{(1+e^{-ap})(1-e^{-ap})} \frac{k}{p} (1-e^{-ap})^2$$

$$= \frac{k\,(1-e^{-ap})}{p\,(1+e^{-ap})}$$

Q. Find $L[|\sin t|]$

mod ki sint ki

since $|\sin t|$ is a periodic with period π π limit

$\cancel{L[|\sin t|]} =$

$$L[f(p)] = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\,dt$$

$$L(|\sin t|) = \frac{1}{1-e^{-\pi p}} \int_0^{\pi} e^{-pt} |\sin t| \, dt$$

$$= \frac{1}{1-e^{-\pi p}} \int_0^{\pi} e^{-pt} \sin t \, dt.$$

$$= \frac{1}{1-e^{-\pi p}} \left[\frac{e^{-pt}}{1+p^2} (-p \sin t - \cos t) \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-\pi p}} \left[\frac{e^{-\pi p}}{1+p^2} (0+1) - \frac{1}{1+p^2} (0-1) \right]$$

$$L(|\sin t|) = \frac{1+e^{-\pi p}}{(1+p^2)(1+e^{-\pi p})}$$